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A Dynamical System Approach to SOC Models of Zhang's Type

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We discuss Zhang's model of SOC in the framework of hyperbolic dynamical systems with singularities. The fractal structure of the invariant energy distribution, correlation decay-like phenomena, and symbolic coding are discussed.

KEY WORDS: Self-organized criticality; hyperbolic dynamical systems with singularities; invariant sets and invariant measures; fractal structure of the invariant energy distribution.

Systems exhibiting self-organized criticality (SOC) reach spontaneously an equilibrium state with scale-invariant characteristics, reminiscent of traditional equilibrium systems at critical point.[1,5] At the moment, no description of these models from a dynamical system point of view has been tempted.

In this letter, we analyze Zhang's model with constant activation energy in a dynamical systems framework and outline some key properties. Addition of energy at a given site induces a piecewise affine transformation on the set of stable configurations, giving the redistribution of energy after the corresponding avalanches. The domains of continuity of this mapping partition the set of stable configurations so that there is a one to one correspondence between the partition elements and the characteristic of the avalanches. We then show how the whole dynamics is described as a *piecewise affine hyperbolic* system with SBR invariant measures. We discuss the question of ergodicity in this mathematical framework. We then give evidences that the support of the invariant energy measure can be *fractal* for certain values of E_c. As a result, we exhibit nontrivial behavior even for *one-dimensional* models, for $E_c/\delta E < 1$ namely, outside the range usually

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considered. The existence of a singularity set implies that the distance between two initial conditions whose trajectories enter eventually two different elements at a given time, increases. This is the only separation effect one has and this gives an explanation of the initial conditions sensitivity observed by Bak.

DESCRIPTION OF THE MODEL

In this paper we deal with Zhang's model on a d-dimensional, connected subgraph $\Lambda \subset \mathbb{Z}^d$, with nearest neighbors edges, though the formalism we develop holds almost word by word for more general graphs.[3] Let $^{\partial}\Lambda$ be the boundary of Λ, namely the set of points in $\mathbb{Z}^d$ at distance 1 from Λ and N the cardinality of Λ. Each site $i \in \Lambda$ is characterized by its energy E_i, which is a nonnegative real number. The "state" of the network is completely defined by the configuration of energies $\mathbf{E} = \{E_i\}_{i \in \Lambda}$. Let E_c be a real, positive number, called the critical energy, and $\Omega = [0, E_c[^N$. A configuration $\mathbf{E}$ is "stable" iff $\mathbf{E} \in \Omega$ and "unstable" or critical otherwise. If $\mathbf{E}$ is stable then we choose a site i at random with probability $1/N$, and add to it energy δE. In the initial model by Zhang[5] the energy quantum δE was not a constant. In this paper we take δE as a constant, fixed to $\delta E = 1$ without loss of generality. If a site i is critical ($E_i > E_c$), its whole energy is relaxed in equal parts to its $2d$ neighbors. The sites of $^{\partial}\Lambda$ have always zero energy (dissipation on the boundaries). An avalanche is characterized by the succession of relaxation events generated by the activation of a given site, until a stable state is reached. Two avalanches are said to be *equivalent* iff they have the same succession of relaxation events, i.e., if the same sites relax at the same time.

PIECEWISE AFFINE MAPPINGS

Addition of energy at site i induces a transformation $\mathcal{T}_i$ on Ω which is either the trivial translation along the direction i (the energy of i increases without relaxation) or gives the stable state obtained after the avalanche. Due to the very definition of the model, $\mathcal{T}_i$ is not a continuous map but Ω is partitioned into a finite number of elements $\mathcal{S}_i^l$, $l \in \{1,..., \mathcal{A}(i)\}$, such that $\mathcal{T}_{i|\mathcal{S}_i^l} \stackrel{\text{def}}{=} \mathcal{T}_i^l$ is continuous. Each element $\mathcal{S}_i^l$ corresponds to a class of equivalent avalanches. Therefore all characteristics of an avalanche are given when knowing the corresponding element $\mathcal{S}_i^l$ and mappings $\mathcal{T}_i^l$. The $\mathcal{T}_i^l$'s are *affine*, i.e., $\mathcal{T}_i^l(\mathbf{E}) = \mathcal{L}_i^l(\mathbf{E} + \delta E \mathbf{e}_i)$, where $\mathcal{L}_i^l$ is linear, and $\mathbf{e}_i$ is the canonical basis vector of $\mathbb{R}^N$, corresponding to the activation of site i. Each element $\mathcal{S}_i^l$ is defined by simple sets of inequalities. For example, the element corresponding to "trivial avalanches" (activation of the site i

without generating any relaxation) is simply given by the inequality $E_i < E_c - 1$; the element corresponding to one step on the left in one dimension by $E_i \geqslant E_c - 1$, $E_{i-1} + E_i/2 < E_c - \frac{1}{2}$, etc....

The important observation is now that each mapping $\mathscr{L}_i^l$ (more precisely its extension to $\mathbb{R}^N$) has only eigenvalues of modulus lower or equal than 1. Indeed, the total energy (L_1 norm) does not increase along the relaxation process and therefore the L_1 norm of $\mathscr{L}_i^l$ is not larger than 1. Furthermore there exists an integer $K(\Lambda, E_c)$ such that any K composition $\mathscr{T}_{i_K}^{l_K} \circ \cdots \circ \mathscr{T}_{i_1}^{l_1}$ has only eigenvalues inside the unit circle.[3] This property comes from the fact that, for any activation sequence, and for any initial condition, the total loss of energy via the boundary has contribution from all sites. Moreover, because the whole energy of a critical site is redistributed proportionally on each neighbors, the total loss of energy on the boundary is bounded from below by a quantity proportional to the total initial energy, with a constant strictly lower than one. This implies that the mapping $\mathscr{T}_{i_K}^{l_K} \circ \cdots \circ \mathscr{T}_{i_1}^{l_1}$ is a contraction for the L_1 norm.

Due to the reset to zero of a critical site after relaxation, each $\mathscr{L}_i^l$'s is a projection onto a subspace of $\mathbb{R}^N$, whose dimension increases with the number of involved sites in the avalanches. Indeed, $\mathscr{L}_i^l$ is a left product of matrices sending the configuration at time t in the avalanche, on the configuration at time $t+1$. These matrices have a number of zero eigenvalues given by the number of sites set to zero at the corresponding time step. Multiplying these matrices gives raise to a kernel whose dimension is the maximal number of sites which have relaxed at the same time.[3]

EXTENDED DYNAMICAL SYSTEM

Now take a half infinite sequence of activation events $\mathbf{i} = i_1 \cdots i_n \cdots$. The succession of avalanches or translations generated by this sequence, starting from a given initial condition $\mathbf{E}$, is given by the succession of elements $\mathscr{S}_{i_k}^{l_k}$, $k = 1 \cdots n \cdots$ visited by the trajectory of $\mathbf{E}$ under the composed mapping $\cdots \circ \mathscr{T}_{i_n} \circ \cdots \circ \mathscr{T}_{i_1}(\mathbf{E})$. In particular, the statistics of avalanche duration, avalanche size, etc..., along this sequence is encoded by the frequency with which the trajectory visits the different elements $\mathscr{S}_{i_k}^{l_j}$.

It is natural to view the process of random activations also as a dynamical System (Σ, σ), namely the full shift σ on the set Σ of the N symbols labeling the sites. The SOC dynamics is then just a skew product system on $\mathscr{M} = \Sigma \times \Omega$, labeled by $(\mathscr{T}, \mathscr{M})$. Furthermore (Σ, σ) is conjugated to the multiplication by $N \bmod 1$, up to a countable number of points where the correspondence is not unique. The advantage of using a smooth representation for the shift system is the possibility of speaking about Lyapunov exponents, smooth elements of stable and unstable manifolds, SBR measures, and so on. In this framework the SOC system is described by piecewise affine mappings, whose tangent space is split into a contracting part, an expanding part (the direction of the full shift), and a kernel. As soon as we are interested in the behavior of sufficiently long sequence, we can restrict the system to the projection onto the complement of the maximal kernel. This kernel is at most $\mathbb{R}^{N-1}$.

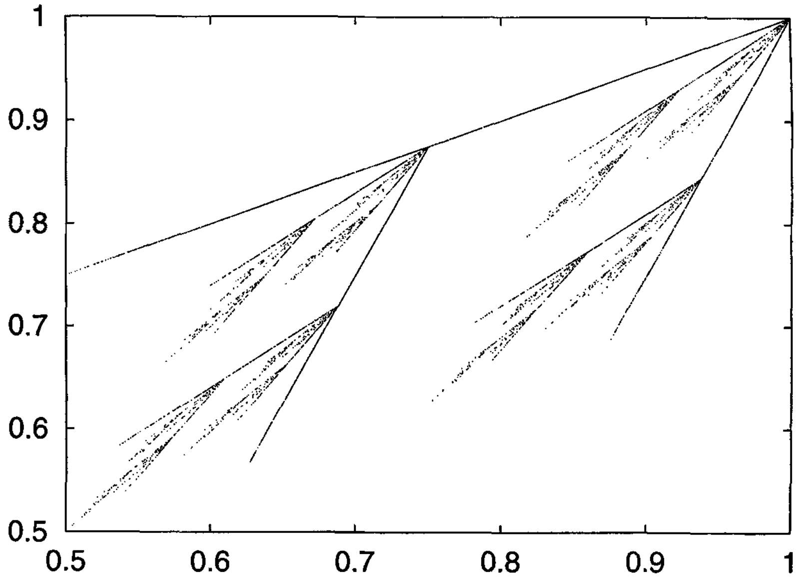
It follows that the previously described class of SOC models, can entirely be studied within the framework of piecewise smooth hyperbolic dynamical systems with singularities.

INVARIANT SETS AND INVARIANT MEASURES

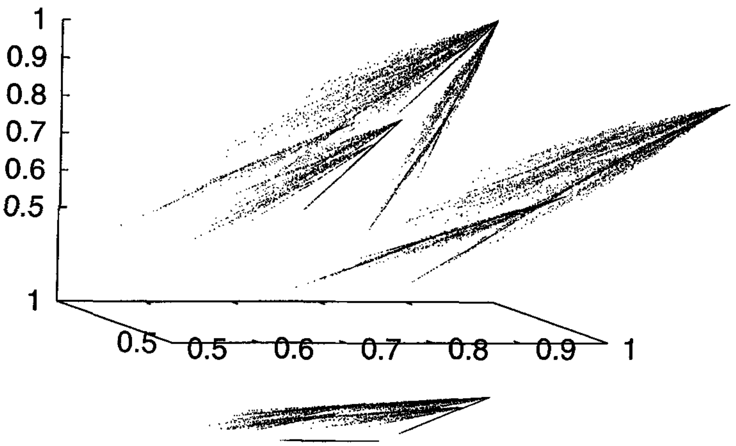
Physically observable convergence to equilibrium corresponds to the convergence of a nonzero Lebesgue measure set of initial conditions towards an invariant set, supporting an invariant measure μ. Therefore the topological and metric properties of the attractors play a key role in the characterization of SOC equilibrium. The characterization of the invariant set from a topological point of view can be best achieved when the system is conjugated to a subshift of finite type (SFT). This means that the trajectory of almost any point can be uniquely encoded in a Markov transition graph between a *finite* number of elements. Unfortunately, this property usually does not hold when the invariant set intersects the singularity set and a complete classification of those E_c values which give raise to an SFT for a fixed Λ seems to be a difficult task. We can show that this property holds in one dimension at least for $E_c \in [1, 2]$. On mathematical ground we conjecture that there exists a sufficiently small E_c^0 such that for almost every $E_c \in [0, E_c^0[$ the system is conjugated to a SFT.

From a physical point of view, the metric aspects are most relevant because the *statistical* properties of avalanche observables are encoded in the frequency with which the trajectory of μ almost all initial condition visits the elements $\mathcal{S}_i^l$. From the subsection above it follows that the unstable manifold of the extended dynamical system admits the Lebesgue measure as an invariant measure. Therefore, the basic objects to characterize the asymptotic orbit structure of the SOC dynamics are the SBR-measures. These are the physically relevant measures because they describe the asymptotic distribution of Lebesgue almost all initial conditions. Note that the quantity usually considered by physicists, namely the invariant distribution of energy is not μ which the invariant measure of the extended system, but rather its *projection* onto Ω, $\mu_{|\Omega}$.

An important quantity to characterize the fine structure of an invariant measure μ is its Haussdorf dimension $HD(\mu)$, which is defined as the infimum of the Haussdorf dimensions of the measure one sets. It



(a)



(b)

Fig. 1. (a) Two dimensional section of the support of the invariant energy measure for $E_c = 1$, $N = 3$. (b) Three dimensional section of the support of the invariant energy measure for $E_c = 1$, $N = 4$.

follows from the general theory of dynamical systems with complete hyperbolic structure (all Lyapunov exponents different from zero) that, when μ is supported on an attractor with a one-dimensional unstable manifold, $HD(\mu)$ becomes arbitrary close to one if the absolute value of the ratio between the largest negative Lyapunov exponent and the largest positive one is sufficiently large. In SOC models, this intuitively means that the contraction rate (determined by the proportion of energy lost on the boundaries, and therefore by E_c) in any stable direction dominates the expansion rate (determined by the number of nodes) sufficiently. The implication for the Zhang's models on a fixed graph is that for any $\varepsilon > 0$ there exists a E_c value (larger than zero, but sufficiently small) such that the Haussdorf dimension of the invariant energy distribution in Ω is less than ε. It is a much harder problem to get explicit bounds on $HD(\mu)$ as a function of Λ and E_c. Analytic computations are possible when the model is conjugated to a subshift of finite type. An explicit computation of $HD(\mu)$ can be made for a 2 sites model. The fractal structure is also numerically revealed, for one-dimensional model with N nodes. In Figs. 1a, 1b we have plotted a two-dimensional section of a trajectory with 100,000 points (100,000 transients) for $E_c = 1$, $N = 3$ and a three-dimensional section for a trajectory of the same duration for $E_c = 1$, $N = 4$. These pictures indicate that the invariant set has a tree like Cantor structure. For larger N this is no longer visible on the projection. A careful analysis of the kernel structure of the composed map in the 3 nodes case explains the global structure in Fig. 1a.[3] This shows that besides the local fractal characteristics of μ there is a non-trivial global geometry of the maximal invariant set which reflects the dependence between the states of the nodes.

Notice that a very special case happens in one dimension, for $E_c \in [1, 2]$. Namely we have a collapse of the measure onto a finite number of points and the set $\mathcal{I} = \{\exists!\ i \text{ s.t. } \forall k \neq i,\ E_k = 1, \text{ and } E_i = 0 \text{ or } 1\}$ is the unique invariant set.

ERGODICITY

Ergodicity is commonly assumed in the papers dealing with SOC. From the mathematical point of view, there are good reasons to believe that ergodicity holds at least for almost every E_c values.[3] For the case of SFT one can always check it algorithmically and in this case it seems that a general proof is accessible. We will discuss the problems involved in the proof of ergodicity elsewhere. Under the assumption of ergodicity, every initial condition, for almost any activation sequence approaches the same invariant set, and the probability to get an avalanche of a given type is just the probability of the corresponding domain of continuity with respect to the invariant measure. By the ergodic theorem this probability is just the frequency of visits of this domain for almost every activation sequence. Therefore, the statistical properties of avalanche in the SOC state are related to the metric properties of the invariant set, and its intersection with the partition elements.

DISTRIBUTION OF AVALANCHES

The difference of the structure of the invariant set according to the value of E_c, suggests that the equilibrium distribution of energy has also a different form. Indeed, for $E_c < 1$ the statistical distribution of avalanches observables exhibits a rather nontrivial structure. For example, the distribution of avalanches duration is shown for various values of E_c: $E_c = 0.1, 0.2, 0.3$ (Fig. 2a), $E_c = 0.5, 0.7, 0.9$ (Fig. 2b) $E_c = 2$ (Fig. 2c).

This difference in behavior comes from the fact that the avalanche shape exhibits a richer and richer variety when E_c decreases. After an avalanche all sites reached by the front have energy $\geqslant E_c/2d$ but one site with a zero value. Therefore, after a sufficiently long sequence of activations such that any site has become critical, all sites have either energy $\geqslant E_c/2d$ or an integer value ($< E_c/2d$). In one dimension, this implies that all avalanches either reach a boundary or a stopping site with integer value. Moreover, for $E_c \geqslant 1$ a zero energy site always stops a front, and the avalanche have all a simple rectangle-like shape. In this case, the statistical properties of avalanches are much easier to get. For $E_c < 1$ the avalanches can either be stopped by a zero ("low" energy configurations), or go through the zero ("high" energy configurations).[3] In the first case the avalanche shapes are identical to the case $E_c \geqslant 1$, while in the second several reflexions on the boundaries occur before the avalanche has lost sufficiently enough energy to be stopped by a zero. The number of possible reflections increases for decreasing E_c.[3]

These results show that various equilibrium states can be encountered according the range of the parameter E_c. This raise the question of what is the "universal" behavior of these models.

SEPARATION EFFECTS

Several people have reported on certain properties of SOC models which seem to indicate that there is a weak form of correlation decay, namely a polynomial one (called by Bak the border of chaos).[2] Due to the expansion along the shift direction, one has trivial exponential decay for the extended system. With respect to the above-mentioned observations the

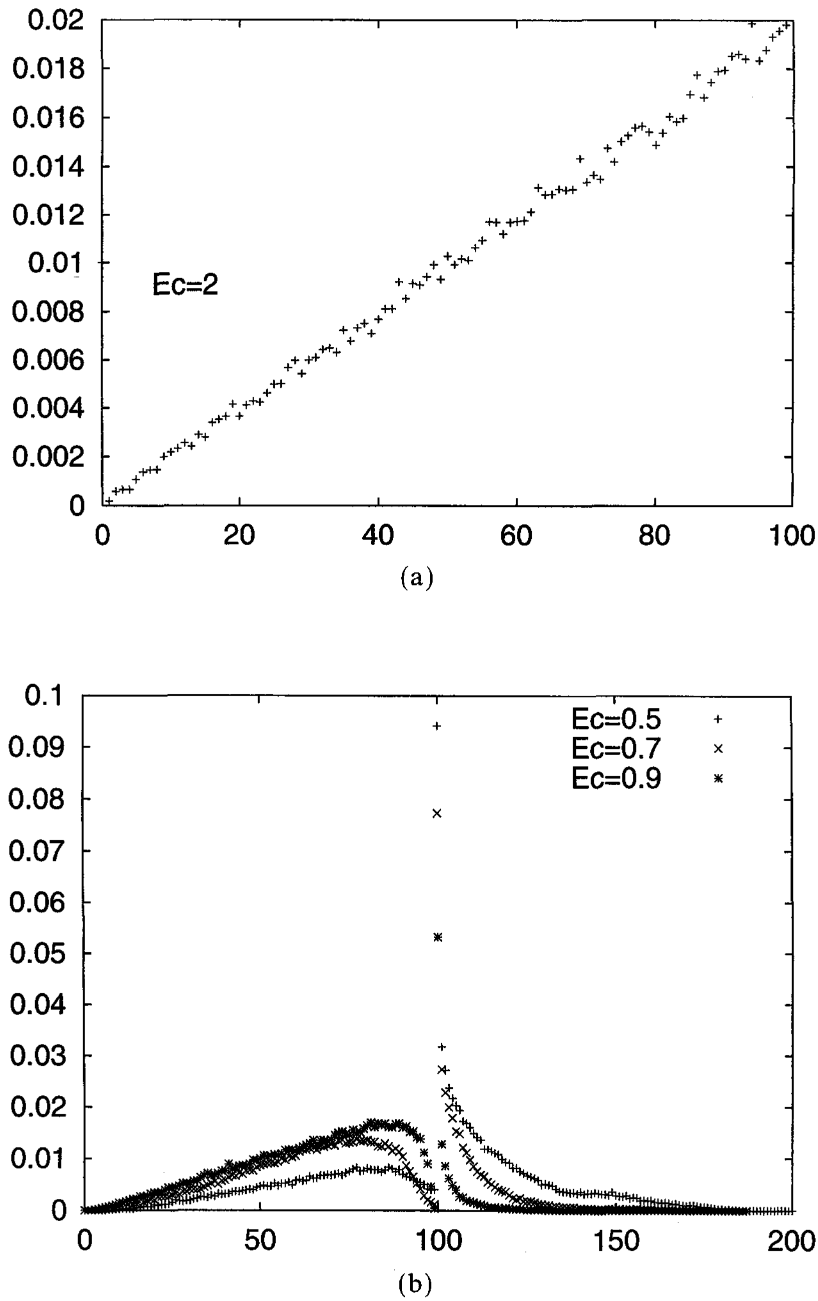
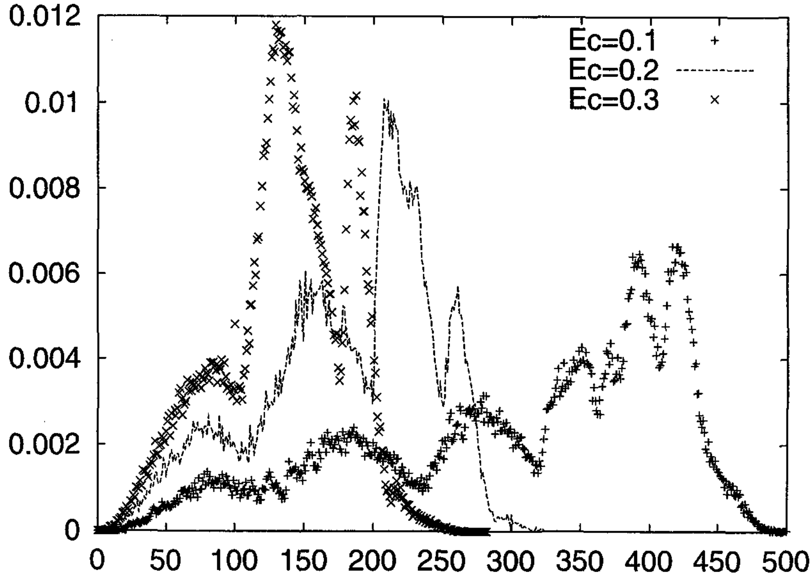


Fig. 2. Distribution of avalanche duration for various value of E_c and $N = 100$.



(c)

Fig. 2 (*Continued*)

situation is different, because there, one fixes an activation sequence and studies the separation of two closeby points from Ω with respect to the trajectories obtained for this fixed activation sequence. Due to the hyperbolic structure, there is no *local* correlation decay in this case, as soon as the invariant measure does not "sit" on the singularity set $\mathscr{S}$. Indeed, two sufficiently close initial conditions following the same activation sequence enter the same elements S_i^l and the distance is contracted to zero by the composed mapping $\cdots \circ \mathscr{T}_{i_n} \circ \cdots \mathscr{T}_{i_1}$. More precisely, if the invariant measure is such that for any sufficiently small ε, $\exists \alpha > 0$, s.t. $\mu(\mathscr{U}_\varepsilon(\mathscr{S})) < C\varepsilon^\alpha$, $C > 0$, where $\mathscr{U}_\varepsilon(\mathscr{S})$ is the ε-neighbourhood of $\mathscr{S}$, one has the following result: for almost every activation sequence $\mathbf{i}$ and almost every point $\mathbf{X} \in \Omega$, $\exists \eta \equiv \eta(\mathbf{X}, \mathbf{i})$ such that for all $\mathbf{Y} \in \mathscr{B}(\mathbf{X}, \eta)$, the distance $d(\mathscr{T}^n(\mathbf{X}, \mathbf{i}), \mathscr{T}^n(\mathbf{Y}, \mathbf{i})) \to 0$ uniformly when $n \to \infty$[3] (this is an easy consequence of the local hyperbolicity and the Borel Cantelli lemma). This condition is also sufficient to establish the existence of a smooth local manifold almost everywhere.

But, what about the observations of Bak? There one has to deal with a *semi local* notion of separation. From what is said above it is clear that separation of $\mathbf{X}$ and $\mathbf{Y}$ along a fixed activation sequence can only happen if **X** and **Y** fall eventually in different domains of continuity $S_{i_1}^{l_1}$, $S_{i_2}^{l_2}$. This can only be the case if **X** and **Y** are not on the same smooth piece of the local stable manifold $\mathscr{W}_{\text{loc}}^S(\mathbf{X})$ of **X** (resp **Y**). Due to the presence of singularities the size of $\mathscr{W}_{\text{loc}}^S(\mathbf{X})$ is not uniform with respect to **X** (an exception is the case where the system is conjugated to a SFT, where one gets uniform bounds on the size of $\mathscr{W}_{\text{loc}}^S(\mathbf{X})$ for **X** being in $\mu_{|\Omega}$). Because we are interested in results with respect to almost every activation sequence the quantity $P(\mathbf{X}, \mathbf{Y}) = \text{Prob}\{(\mathbf{Y}, \mathbf{i}) \notin \mathscr{W}_{\text{loc}}^S(\mathbf{X})\}$, where the probability is taken with respect to **i** and where $\mathbf{X}, \mathbf{Y} \in \Omega$, is exactly what measures the probability of observing separation. Several refinements of this notion are possible, which take into account bounds on the time when the separation happens (see ref. 3). Because one wants to get $P(\mathbf{X}, \mathbf{Y})$ mainly as a function of $d(\mathbf{X}, \mathbf{Y})$ all what one has to study is the distribution of the size of $\mathscr{W}_{\text{loc}}^S(\mathbf{X})$. This is a standard problem in dynamical systems theory, especially for the class of hyperbolic systems with singularities where far reaching methods are available.[4] We will discuss this aspect in details in a forthcoming paper.[3]

A SIMPLE EXAMPLE

The most simple example (2 nodes) admits already a rich dynamical structure. Of course with respect to the classical questions about distributions of avalanche sizes, there is nothing nontrivial to say, but besides this the measure structure and the symbolic dynamics exhibit already much of the rich behavior in the many node case. For a two nodes model, where one of the node is always zero, we can replace Ω by the interval $[0, E_c[$. We then denote by $\mathscr{T}_1$ (resp. $\mathscr{T}_2$) the mapping obtained after activation of the nonzero site (resp. the zero site. As an example take $E_c = \frac{5}{6}$, the mappings are: $\mathscr{T}_1 = \{\mathscr{T}_1^1 = X/2 + \frac{1}{2}$ if $X < \frac{2}{3}$ and $\mathscr{T}_1^2 = X/4 + \frac{1}{4}$ otherwise and $\mathscr{T}_2 = X/2 + \frac{1}{4}$ for $X > \frac{1}{3}$, X being the value of the nonzero site. The following arrow sequence describes which consecutive applications of the mappings are possible in the asymptotic case: $\mathscr{T}_1^1 \to \mathscr{T}_1^2$, $\mathscr{T}_2$; $\mathscr{T}_1^2 \to \mathscr{T}_1^1$, $\mathscr{T}_2$; $\mathscr{T}_2 \to \mathscr{T}_1^1$, $\mathscr{T}_2$, each legal transition occurring with probability $\frac{1}{2}$. The maximal invariant set is contained in the interval $[\frac{1}{3}, \frac{5}{6}]$ but the Haussdorf dimension of $\mu_{|\Omega}$ is *less than one*. Indeed, from the above transition graph, it follows that the probability to find $\mathscr{T}_1^2$ in a typical activation sequence is $\frac{1}{6}$, hence the negative Lyapunov exponent (in base 2 logarithm) is $-\frac{7}{6}$. The system being invertible on the invariant set, the Young formula $HD(\mu) = h_\mu(\mathscr{T})(1/|\lambda^+| + 1/|\lambda^-|)$, gives $HD(\mu_{|\Omega}) = \frac{6}{7}$. Here $\lambda^\pm$ are the positive (resp negative) Lyapunov exponent, and h_μ the Kolmogorov Sinai entropy which is equal to $|\lambda^+|$ in the SBR case.

DISCUSSION

The approach presented in this paper allows to discuss SOC in the framework of the theory of hyperbolic systems with singularities. We list now several natural questions.

— Stability of the hyperbolic structure. The important point is to keep the local contraction property, which is robust with respect to changes in the distribution of energy in the relaxation process, even if the whole energy is not distributed uniformely to the neighbors. This implies that models where only a fixed value of energy is redistributed have different properties. Analytical generalization, where one smooth out the threshold condition, are still good candidates for hyperbolic behavior though certainly not uniform.

— Stability under stochastic perturbations. In this case (e.g., when δE is a random variable), we expect stochastic stability at least in the case where the nonperturbed system is structurally stable.

— Dynamics on subgraphs on $\mathbb{Z}^d$ for $d > 1$. Of course, one can not expect to avoid in the higher dimensional case the standard problems one is faced in SOC-dynamics, which are mainly of combinatorial nature. Even with full knowledge of the invariant distribution, there remains to estimate the numbers of domains $\mathcal{S}_i^l$ which give raise to avalanches of the same size. This is in principle algorithmically computable, but probably difficult to achieve in an effective way. The combinatorial difficulty problem are likely to become less important if E_c becomes sufficiently small.

— Relationship between the characteristics of the invariant measure and the critical exponents for the distribution of avalanches observables. The multifractal spectrum is a much finer quantity to characterize the invariant distribution than the Haussdorf dimension, though it is in general difficult to establish its existence. Because it characterizes the local variation of density of the invariant measure, one might expect close relationships between the scaling exponents of the multifractal spectrum, and the critical exponents computed in SOC.

The interesting question about universality in the thermodynamic limit where we lose the hyperbolic structure will be discussed in a forthcoming paper.[3]

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